THE UNIVERSITY OF ALBERTA

HOMEOTOPY GROUPS OF 2-MANIFOLDS

by

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ABSTRACT

The main purpose of this thesis is to compute the homeotopy and isotopy groups of various spaces of homeomorphisms of orientable 2-manifolds with boundary. The space $H(M)$ of homeomorphisms of the manifold M is given the compact open topology under which it forms a topological transformation group on M. The homeotopy group of the manifold M is defined to be the isotopy classes of $H(M)$ and we similarly define the subhomeotopy group for a subspace S of $H(M)$ which is called the isotopy group of S. Further, for $n \geq 1$, we consider the nth homotopy group of the space of homeomorphisms. In Chapter I we prove the preliminary fundamental theorems about the isotopy relations of homeomorphisms on 2-manifolds with boundary.

In Chapter II we calculate the homeotopy and isotopy groups for the orientable 2-manifolds with boundary holes. By extending the concept of the winding number defined by H. Gluck, for the manifold M obtained from the 2-sphere by removing the interiors of three disjoint subdisks, we show that the isotopy group of the subspace of homeomorphisms fixing $Bd(M)$ pointwise is the direct product of three copies of the integers, and the homeotopy group of M is the direct product of the group of order 2 and the symmetric group on three letters. For the orientable 2-manifold M_n with n boundary holes, it is shown that the homeotopy group is homomorphic to the group $S_n \times J_2$ with kernel $\mathcal{J}[H_n^+(M_n)]$, the isotopy group of the subspace of the homeomorphisms which fix n boundary curves setwise with orientations preserved.

In Chapter III we consider homeomorphisms of the manifold M_n obtained from the 2-sphere by removing the interiors of n disjoint subdisks. The isotopy groups of various subspaces of homeomorphisms of M_n are represented as the subgroups of automorphisms of the homotopy group $\pi_1(M_n, x_o)$.

Chapter IV deals with the isotropy groups at an interior point. By using the homeotopy exact sequence, we calculate the isotopy and nth homeotopy groups of the isotropy group for certain orientable 2-manifolds.

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DEDICATION

This dissertation is dedicated

to the memory of my parents.

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CHAPTER 0

INTRODUCTION

One of the main problems in algebraic topology has been to classify and characterize spaces by means of topological invariants which are algebraic in character. A progress toward the goal will require new algebraic invariants which are not homotopy invariants. Related to this is certainly the homeomorphism group $H(X)$ of a space X onto itself. As might be expected, the compact open topology is a natural choice for $H(X)$. In fact, Arens [3] showed that if X is locally compact, locally connected and Hausdorff, then the compact open topology is the smallest topology on $H(X)$ in which $H(X)$ forms a topological transformation group of X. Thus, in what follows $H(X)$ will always be given the compact open topology.

Let $I = [0,1]$. For $h_0, h_1 \in H(X)$, we define h_0 to be isotopic to h_1 (denoted by $h_0 \approx h_1$) if and only if there is a continuous function $G : X \times I \to X$ such that (i) $G(x,0) = h_0(x)$, (ii) $G(x,1) = h_1(x)$ and (iii) for each fixed $t \in I$, the function $G_t \in H(X)$, where G_t is defined by $G_t(x) = G(x,t)$ for all $x \in X$. The homeotopy group of X is defined to be the group of the isotopy classes of $H(X)$, which will be denoted by $\mathcal{H}(X)$ and $[h]$ will denote the isotopy class of a homeomorphism h. It can be redefined equivalently as follows. Let $H_o(X)$ denote the arc-component of the identity e in $H(X)$. Then $H_o(X)$ is a normal subgroup of $H(X)$ and the homeotopy group $\mathcal{H}(X) = H(X)/H_o(X)$ is the group of arc-components of $H(X)$. The equivalence relation defined by $H_o(X)$

is also isotopy.

We can also define the isotopy relation in a subspace $H'(X)$ of $H(X)$ and the subhomeotopy group in a similar way, which we will distinctively call the isotopy group of $H'(X)$. For $x \varepsilon X$, the isotropy group at x will be defined to be $H(X,x) = \{h \varepsilon H(X) | h(x)=x\}$. The nth homeotopy group of X is defined to be the nth homotopy group of the space of homeomorphisms. We note that $\pi_0[H(X)] = H(X)/H_o(X) = \mathcal{H}(X)$. The homeotopy groups are thus topological invariants (up to isomorphism) of a given space X and homotopy invariants of the homeomorphism group of X.

In the main part of this thesis we shall assume for simplicity that the manifolds are given a piecewise linear structure, and all homeomorphisms, isotopies, imbeddings of arcs, etc., are in the piecewise linear category. This is in fact no real restriction, since it follows from ([6], Appendix) that the results also hold in the general topological category.

A brief history of the present work is, to the best of our knowledge, as follows. J will denote the group of integers, J^k the direct product of k copies of J, and J_2 the integers mod 2. In 1914, Tietze showed that the homeotopy group of the 2-sphere is J_2[23]. This was proven again by Kneser in 1926 [14], Baer in 1928 [4], Schreier and Ulam in 1934 [21], and most recently by Fisher in 1960 [7]. In [14] Kneser also obtained a result that the homeotopy group of the disk is J_2. Tietze [23] and Smith [22], in 1914 and 1917 respectively, proved that a homeomorphism of the disk which leaves the boundary

pointwise fixed is homotopic to the identity through a family of homeomorphisms which leave the boundary pointwise fixed. In 1917, Veblen [24] proved that a homeomorphism of the closed n-cell which leaves the boundary pointwise fixed is homotopic to the identity through a family of imbeddings of the closed n-cell onto n-space which do not necessarily map the n-cell onto itself. This latter result was modified by Alexander who, in 1923, proved that the space of homeomorphisms of an n-cell onto itself leaving the boundary pointwise fixed is contractible [2]. This result has been a most important tool for further development in this area of study. In 1958, Hamstrom and Dyer proved that the space of homeomorphisms of the closed annulus onto itself leaving one of the boundary holes pointwise fixed is contractible [9]. In 1962, in terms of the winding number of a homeomorphism of the annulus, Gluck proved that the isotopy group of homeomorphisms of the closed annulus onto itself leaving the boundary pointwise fixed is J [8]. He also showed that the homeotopy group of the annulus is $J_2 \times J_2$. In 1962, Hamstrom proved that the identity component of the space of homeomorphisms on a disk with holes leaving its boundary pointwise fixed is homotopically trivial [10], and in 1965, showed a corresponding result for a torus [11]. In 1966, Hamstrom also proved that if M is a compact orientable surface with boundary and two or more handles and S is the identity component of the space of homeomorphisms of M onto itself which leave $Bd(M)$ pointwise fixed, then S is homotopically trivial [12]. In 1962, Lickorish observed that any orientation preserving homeomorphism h of a closed connected orientable 2-manifold M is isotopic to the product of a sequence of homeomorphisms supported on annuli in M [15].

In 1964, he produced a finite collection of annuli on a closed orientable manifold M, whose corresponding homeomorphisms generate all orientation preserving homeomorphisms of M [16]. In 1968, Quintas listed in [20] the known results along these lines as well as the unsolved open problems some of which have been solved in this thesis.

Basic Notations

M_n will denote an orientable 2-manifold with n boundary holes.

$H^+(M_n) = \{h \in H(M_n) \mid h$ is orientation preserving on $M_n\}$

$H^-(M_n) = \{h \in H(M_n) \mid h$ is orientation reversing on $M_n\}$

$H_m(M_n) = \{h \in H(M_n) \mid h(C_i) = C_i$ for certain m boundary curves $C_i\}$

$H^t(M_n) = \{h \in H(M_n) \mid h = e$ on certain t boundary curves$\}$

$H_m^+(M_n) = H^+(M_n) \cap H_m(M_n)$

$H_m^t(M_n)$ is similarly defined for certain m and t boundary curves where $m + t \leq n$ and the m holes are different from the t holes. Let $H'(M_n)$ be a subspace of $H(M_n)$. Then $\mathcal{J}[H'(M_n)]$ will denote the isotopy group generated by the isotopy classes of $H'(M_n)$ classified by the isotopy paths in the same space $H'(M_n)$. If the isotopy paths are taken in a different space G, then we denote by $\mathcal{J}_G[H'(M_n)]$ the isotopy group generated by the isotopy classes of $H'(M_n)$. S_n will denote the symmetric group on n letters.

In Chapter I the preliminary fundamental theorems are given for use in the later chapters. We prove that if $\{\alpha_i\}_{i\in I}$ is a finite collection of arcs in a 2-manifold M with boundary such that

(i) each α_i connects boundary points with $\mathrm{Int}(\alpha_i) \subset \mathrm{Int}(M)$ for

for each $i \in I$, (ii) $\alpha_i \cap \alpha_j = \phi$ for $i \neq j$ and (iii) cutting M along the α_i's leads to a disk, then a homeomorphism h such that $h = e$ on $Bd(M)$ and $h(\alpha_i) \simeq \alpha_i$ for each $i \in I$, is isotopic to the identity on M by an isotopy fixing $Bd(M)$ pointwise. It is also shown that if M_n is an orientable 2-manifold with n boundary holes and h a homeomorphism in $H_n(M_n)$, then h must have the same orientation on all the boundary curves.

In Chapter II we calculate the homeotopy and isotopy groups for the orientable 2-manifolds with boundary holes. By extending the concept of the winding number defined by Gluck [8], for the manifold M obtained from the 2-sphere by removing the interiors of three disjoint subdisks, it is shown that the isotopy group of the subspace of homeomorphisms fixing $Bd(M)$ pointwise is $J \times J \times J$, and the homeotopy group $\mathcal{H}(M)$ is $S_3 \times J_2$. We also calculate the isotopy groups of various subspaces of $H(M)$. For the orientable 2-manifold M_n with n boundary holes, it is shown that the homeotopy group $\mathcal{H}(M_n)$ is homomorphic to the group $S_n \times J_2$ with the kernel $\mathcal{J}[H_n^+(M_n)]$. Similar results are obtained for the subspaces $H_m(M_n)$ and $H_m^t(M_n)$.

In Chapter III we consider homeomorphisms of the manifold M_n obtained from the 2-sphere by removing the interiors of n disjoint subdisks. The isotopy groups of various subspaces of $H(M_n)$ are represented as the subspaces of $\mathcal{A}[\pi_1(M_n, x_o)]$, the group of automorphisms of the homotopy group $\pi_1(M_n, x_o)$ with $x_o \in Bd(M_n)$.

Finally Chapter IV deals with the isotropy group at an

interior point. By using the homeotopy exact sequence, we calculate the isotopy and nth homeotopy groups of the isotropy group for certain orientable 2-manifolds. It is shown that if D is a disk and $a, x \in \mathrm{Int}(D)$, then $\mathcal{I}[H(D-a,x)] = J_2$ and $\pi_n[H(D-a,x)] = 0$ for $n \geq 1$.

CHAPTER I

ISOTOPY RELATIONS

In this chapter we give some fundamental theorems which will be needed in the sequel. In particular theorems about the isotopy relations and orientations of the homeomorphisms of an orientable 2-manifold are considered.

<u>Lemma 1.1</u> (Alexander [2]). The space of homeomorphisms of a closed n-cell onto itself which leave the boundary of the n-cell pointwise fixed is contractable.

<u>Definition 1.2</u>. An isotopy of a topological space X is a collection $\{G_t\}$, $0 \leq t \leq 1$, of homeomorphisms of X onto itself such that the mapping $G : X \times I \to X$ defined by $G(x,t) = G_t(x)$ is continuous. An isotopy which moves no point on $Bd(X)$ is called a B-isotopy. $h \underset{B}{\approx} g$ will denote that h is B-isotopic to g.

<u>Definition 1.3</u>. An isotopy $\{G_t\}$, $0 \leq t \leq 1$, is called invertible if the collection $\{G_t^{-1}\}$, $0 \leq t \leq 1$, of the inverse homeomorphisms is also an isotopy.

<u>Lemma 1.4</u> (Crowell [5]). Every isotopy G_t, $0 \leq t \leq 1$, of a locally compact Hausdorff space is invertible.

<u>Definition 1.5</u>. The imbeddings f_0, $f_1 : X \to Y$ are ambient isotopic if there is a homeomorphism $G : Y \times I \to Y \times I$ which commutes with projection on I (that is, is level preserving) and has the properties that $G(y,0) = (y,0)$ for all $y \in Y$ and $(f_1(x),1) = G(f_0(x),1)$ for all $x \in X$.

Lemma 1.6 (Epstein [6]). Let M be a 2-manifold with boundary. Let α and β be two arcs in M such that

$$\mathrm{Bd}(M) \cap \alpha = \mathrm{Bd}(\alpha) = \mathrm{Bd}(\beta) = \mathrm{Bd}(M) \cap \beta,$$

and which are homotopic keeping the end points fixed. Then they are ambient isotopic by a B-isotopy.

Lemma 1.7. Let M_n be a 2-manifold with n boundary holes and α be an arc connecting the boundary points with $\mathrm{Int}(\alpha) \subset \mathrm{Int}(M_n)$, and let h and g be any homeomorphisms in $H^n(M_n)$ such that the closed arcs $h(\alpha) \circ \alpha^{-1}$ and $g(\alpha) \circ \alpha^{-1}$ belong to the same homotopy class in $\pi_1(M_n, x_o)$ where x_o is the base point on α. Then $h(\alpha) \simeq g(\alpha)$ with the end points of the arc α held fixed.

Proof: Observe that if $g_1, g_2 \in H^n(M_n)$ and $g_1(\alpha) \circ \alpha^{-1}$ and $g_2(\alpha) \circ \alpha^{-1}$ belong to the same homotopy class in $\pi_1(M_n, x_o)$, then $g_2^{-1} g_1(\alpha) \simeq \alpha$. Thus it is sufficient to consider a homeomorphism $h \in H^n(M_n)$ such that $h(\alpha) \circ \alpha^{-1} \simeq 0$ with x_o as the base point.

Assuming that $h(\alpha) \circ \alpha^{-1} \simeq 0$, we show that $h(\alpha) \simeq \alpha$ with the end points a and b held fixed. Let $\delta, \tau : I \to M_n$ be the paths representing $h(\alpha)$ and α with $\delta(0) = \tau(0) = a$ and $\delta(1) = \tau(1) = b$. We must construct a homotopy $H : I \times I \to M_n$ such that, for every $s, t \in I$, $H(s,0) = \delta(s)$, $H(s,1) = \tau(s)$ and $H(0,t) = a$, $H(1,t) = b$. Then the closed arc $h(\alpha) \circ \alpha^{-1}$ is represented by the loop

$$f(s) = \begin{cases} \delta(2s) \; ; & 0 \leq s \leq \frac{1}{2} \\ \tau(2-2s); & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Since $h(\alpha) \circ \alpha^{-1} \simeq 0$, there exists a map $F : I \times I \to M_n$ such that, for every $s, t \in I$, $F(0,t) = a$, $F(1,t) = a$ and $F(s,t) = f(s)$, $F(s,1) = a$.

Now let q be the point $(\frac{1}{2},0)$ in $I \times I$. Then the line L_t through q with angle of inclination $(1-t)\pi$ meets the boundary of $I \times I$

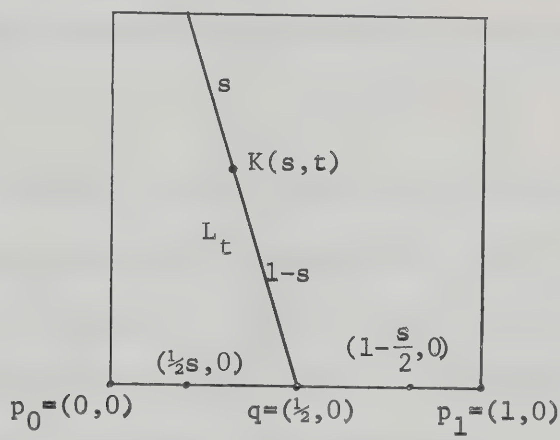

Figure 1

in q and another point P_t if $0 < t < 1$. Let $P_0 = (0,0)$ and $P_1 = (1,0)$. Define a homotopy $K : I \times I \to I \times I$ by taking $K(s,t)$ to be the point which divides the segment $P_t q$ in the ratio $s : (1-s)$.

Letting $H = FK$, we see that, for every $s, t \in I$,

$$H(s,0) = FK(s,0) = F(\tfrac{s}{2},0) = f(\tfrac{s}{2}) = \delta(s)$$

$$H(s,1) = FK(s,1) = F(1-\tfrac{s}{2},0) = f(1-\tfrac{s}{2}) = \tau(s)$$

$$H(0,t) = FK(0,t) = \left\{ \begin{array}{c} F(0,t') \\ F(s',1) \\ F(1,t') \end{array} \right\} = a \quad \text{for every } s', t' \in I$$

$H(1,t) = FK(1,t) = F(\frac{1}{2},0) = f(\frac{1}{2}) = \delta(1) = \tau(1) = b$. Thus $\delta \simeq \tau$ with $\delta(0) = \tau(0) = 0$ and $\delta(1) = \tau(1) = b$. Hence $h(\alpha) \simeq \alpha$ with their end points held fixed.#

Theorem 1.8. Let $\{\alpha_i\}_{i\varepsilon I}$ be a finite collection of arcs in a 2-manifold with boundary such that (i) each α_i connects boundary points with $\mathrm{Int}(\alpha_i) \subset \mathrm{Int}(M)$ for each $i \varepsilon I$, (ii) $\alpha_i \cap \alpha_j = \phi$ for $i \neq j$ and (iii) cutting M along the α_i's leads to a disk. Then if a homeomorphism h has the properties that $h = e$ on $\mathrm{Bd}(M)$ and $h(\alpha_i) \simeq \alpha_i$ for each $i \varepsilon I$, then h is B-isotopic to the identity.

Proof: We prove the theorem by an induction on the number of the arcs α_i.

First we prove it for the case $n = 1$. Let α be an arc satisfying the above conditions in a manifold M and h be a homeomorphism of M onto itself such that $h = e$ on $\mathrm{Bd}(M)$ and $h(\alpha) \simeq \alpha$. Since the two arcs $h(\alpha)$ and α are ambient isotopic by a B-isotopy by Lemma 1.6, there is an isotopy $G_t : M \to M$, $0 \leq t \leq 1$, such that $G_t = e$ on $\mathrm{Bd}(M)$, $G_0 = e$ on M and $G_1^{-1}h = e$ on α. Thus by cutting M along α, we realize that $G_1^{-1}h$ is a homeomorphism of M' such that $G_1^{-1}h = e$ on $\mathrm{Bd}(M')$, where M' is the disk obtained from M by cutting along α. Hence Lemma 1.1 implies that $G_1^{-1}h$ is B-isotopic to the identity on M'. Thus h is also B-isotopic to the identity on M.

Now assuming that the theorem is true for the case $n = k$, we prove it for $n = k + 1$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}\}$ be a collection of the arcs in a 2-manifold M satisfying the above conditions and h be a homeomorphism of M onto itself such that $h = e$ on $\mathrm{Bd}(M)$ and $h(\alpha_i) \simeq \alpha_i$ for $1 \leq i \leq k + 1$. Then in particular we have $h(\alpha_{k+1}) \simeq \alpha_{k+1}$ keeping the end points held fixed and these two arcs are ambient isotopic by a B-isotopy. There is thus a B-isotopy

$H_t : M \to M$, $0 \leq t \leq 1$, such that $H_0 = e$ on M and $H_1^{-1}h = e$ on α_{k+1}. Now cutting M along α_{k+1}, we see that $H_1^{-1}h$ is a homeomorphism of M' such that $H_1^{-1}h = e$ on $Bd(M')$ and $H_1^{-1}h(\alpha_i) \simeq \alpha_i$ for $1 \leq i \leq k$, where M' is a 2-manifold obtained from M by cutting along α_{k+1}. By our assumption $H_1^{-1}h$ is B-isotopic to the identity on M'. Thus h is also B-isotopic to the identity on M and the theorem follows for any integer $n \geq 1$.#

H. Gluck [8] defined the winding number for a homeomorphism of the annulus onto itself which fixes the boundary pointwise as follows.

Let η be the isomorphism of $\pi_1(S^1,0)$ with J which takes the class of the path $f(t) = t$ onto 1. Let α be any path in $S^1 \times I$ from $(0,0)$ to $(0,1)$. Let $P_1 : S^1 \times I \to S^1$ be the natural projection. Then $P_1(\alpha)$ is a closed path in S^1 based at 0. Hence $[P_1(\alpha)]$ is an element of $\pi_1(S^1,0)$ and $\eta([P_1(\alpha)]) = \omega(\alpha)$ is an integer, called the winding number of the path α. It is shown that if h is a homeomorphism of the annulus onto itself which fixes the boundary pointwise, then the number $\omega(h\alpha) - \omega(\alpha)$ is independent of the path α [8].

Definition 1.9. Let h be a homeomorphism of the annulus $A = S^1 \times I$ onto itself which fixes the boundary pointwise, and α be a path in A from $(0,0)$ to $(0,1)$. Then the integer $W[h;A] = \omega(h\alpha) - \omega(\alpha)$ is called the winding number of h on A.

Definition 1.10. Let M be a 2-manifold and A be an annulus in $Int(M)$. Then there is a homeomorphism h_A of the annulus A onto itself such that $W[h_A;A] = 1$ and $h_A = e$ on $Bd(A)$. This

homeomorphism can be extended to M by the identity on $M - A$. We call the extended homeomorphism an A-homeomorphism and denote it by h_A also.

Theorem 1.11. Let A and A' be two annuli which can be deformed to each other on an orientable 2-manifold M. Then two A-homeomorphisms h_A and $h_{A'}$ are B-isotopic to each other.

Proof: Since A can be deformed to A', the internal boundary curves C and C' of A and A' respectively are isotopic. Thus there is an isotopy $G_t : M \to M$, $0 \leq t \leq 1$, such that $G_0 = e$ on M and $G_1(C) = C'$. Hence $G_t h_A G_t^{-1}$, $0 \leq t \leq 1$, is a B-isotopy between the homeomorphisms h_A and $G_1 h_A G_1^{-1}$. Observing that $G_1 h_A G_1^{-1}$ is supported on some annulus A'' in $\mathrm{Int}(M)$ which has the common interval boundary curve C' with the annulus A', it can be shown that $h_{A'}$ is B-isotopic to $G_1 h_A G_1^{-1}$. We note that the winding numbers $W[h_{A'};A']$ and $W[G_1 h_A G_1^{-1};A'']$ are both one. Let A''' be an annulus such that $A''' \subset A' \cap A''$. Then $W[h_{A'''};A'] = W[h_{A'''};A''] = 1$, since the homeomorphism $h_{A'''}$ has the winding number $W[h_{A'''};A'''] = 1$ and is supported on each of the annuli A' and A''. Thus Theorem 1.8 implies that $G_1 h_A G_1^{-1} \underset{B}{\approx} h_{A'''} \underset{B}{\approx} h_{A'}$ and thus $G_1 h_A G_1^{-1} \underset{B}{\approx} h_{A'}$. Hence h_A is B-isotopic to $h_{A'}$.#

Lemma 1.12 (Hamstrom and Dyer [9]). The space of homeomorphisms of the closed annulus onto itself, which leave one of the boundary curves pointwise fixed, is contractible.

Lemma 1.13. Let h be a homeomorphism in $H_n^+(M_n)$. Then h can be deformed to a homeomorphism k such that $k = e$ on $\mathrm{Bd}(M_n)$, where the isotopy path is taken in $H_n^+(M_n)$.

Proof: Define an annulus A_i around each of the boundary curves C_i so that each C_i forms a boundary curve of the annulus A_i, $A_i \subset \mathrm{Int}(M_n) \cup C_i$ and $A_i \cap A_j = \phi$ for $1 \leq i \neq j \leq n$. Now we construct a homeomorphism g;

$$g = \begin{cases} h & \text{on } \mathrm{Bd}(M_n) \\ e & \text{on } M_n - \bigcup_{i=1}^{n} A_i . \end{cases}$$

Then by Lemma 1.12, $g \mid A_i$ is isotopic to the identity on A_i for $1 \leq i \leq n$, since the isotopy is allowed to move on each boundary curve C_i, and hence g is isotopic to the identity on M_n by an isotopy path in $H_n^+(M_n)$. Thus it is clear that hg^{-1} is isotopic to the homeomorphism h in $H_n^+(M_n)$ with $hg^{-1} = e$ on $\mathrm{Bd}(M_n)$. Letting $k = hg^{-1}$, the proof is complete.

Lemma 1.14. Let K and K' be arbitrary triangulations of the same surface or of two homeomorphic surfaces. Then if one of these triangulations is orientable, the other is also orientable.

Proof: This is Theorem 4.13 of [1].

Lemma 1.15 (Epstein [6]). Let h be a homeomorphism of a 2-manifold M onto itself which fixes a base point in M. Then there is an ambient isotopy which is fixed on the base point and which changes h to a piecewise linear homeomorphism.

Lemma 1.16. Let M be a closed orientable 2-manifold and h be a homeomorphism of M onto itself. Then h must have the same orientation on every open subdisk in M.

Proof: By Lemma 1.15 we can assume that M is given a piecewise linear

structure and h a piecewise linear homeomorphism. Let O and O' be any two disjoint open subdisks in M and K be an orientable triangulation of M such that triangles T and T' in K are contained in O and O' respectively. Thus by Lemma 1.14, h has the same orientation on both T and T'. Thus h must have the same orientation on the subdisks O and O'. #

Theorem 1.17. Let M_n be an orientable 2-manifold with n boundary holes and h be a homeomorphism in $H_n(M_n)$. Then h must have the same orientation on all the boundary curves.

Proof: We assume $n > 1$ since otherwise the theorem is trivial.

First we observe that h can be extended to the interiors O_i of the boundary holes C_i for $1 \leq i \leq n$. Let C_i be a boundary hole of M_n. Let c be the center of the hole C_i and a be an arbitrary point on the boundary curve C_i. Denote by $[c,a]$ the line segment between c and a. Then we can obtain an extension h_i of h on $M_n \cup O_i$ by defining as follows. $h_i \equiv h$ on M_n and $h_i([c,a]) = [c,h(a)]$ on $O_i \cup C_i$ with $d(c,P) = d(c,h_i(P))$ for any $P \in [c,a]$.

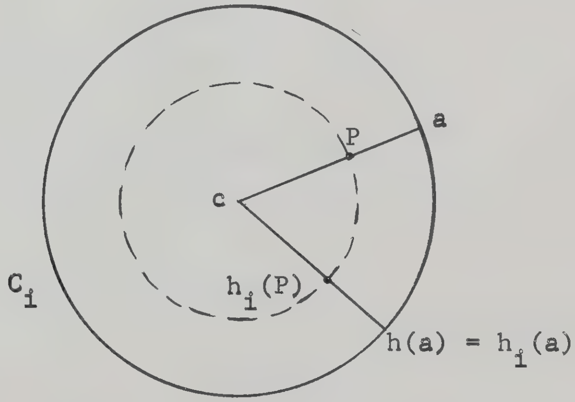

Figure 2

In a similar manner, h can be extended to each of the interiors of all the boundary holes.

Now to prove the theorem, we assume that h is orientation preserving on a boundary curve C_j and reversing on C_k where $1 \leq j \neq k \leq n$. Let $\overline{M}_n$ be a closed orientable 2-manifold obtained from M_n by filling in the interiors of all the boundary holes, and $\overline{h}$ be the extension of the homeomorphism h to $\overline{M}_n$. Let O_j and O_k be the interiors of the boundary holes C_j and C_k respectively. Then it can be seen that $\overline{h}$ must have different orientations on O_j and O_k, since $\overline{h}$ has different orientations on $Bd(\overline{O}_j) = C_j$ and $Bd(\overline{O}_k) = C_k$. Thus we have a contradiction to Lemma 1.16 and h must be orientation preserving or reversing on all the boundary curves. #

<u>Remark 1.18</u>. By Theorem 1.17 we can see that $H^t(M_n) \subset H^+(M_n)$ for any integer $t > 0$.

CHAPTER II

HOMEOTOPY GROUPS

In this chapter we calculate the homeotopy and isotopy groups of the orientable 2-manifold with boundary.

We extend the concept of the winding number defined in Chapter I to the 2-sphere with more than two boundary holes. In what follows we denote by A_i an annulus in $\mathrm{Int}(M_n)$ enclosing only one corresponding boundary hole C_i and h_{A_i} will denote an A_i-homeomorphism of M_n supported on A_i for $1 \leq i \leq n$. By saying that A_i encloses one hole C_i, we mean that A_i can be shrunk onto the boundary curve C_i and thus A_i can be extended up to C_i so that C_i forms a boundary curve of the extended annulus $\overline{A}_i$ while the other boundary remains fixed.

<u>Lemma 2.1</u>. The homeomorphism h_{A_i} $(1 \leq i \leq n)$ is isotopic to the identity by an isotopy path in $H_n^+(M_n)$.

<u>Proof</u>: Let $\overline{A}_i$ be the extended annulus of A_i to the boundary curve C_i. Then $h_{A_i} = e$ on $M_n - \overline{A}_i$ and the isotopy in $H_n^+(M_n)$ is allowed to move on the curve C_i. Thus Lemma 1.12 implies that $h_{A_i} | \overline{A}_i$ is isotopic to the identity by an isotopy which moves on C_i and is fixed on the other boundary of the annulus $\overline{A}_i$. Hence h_{A_i} is isotopic to the identity on M_n in $H_n^+(M_n)$.#

<u>Lemma 2.2</u> (Lickorish [15]). Let M_n be a manifold obtained from the 2-sphere by removing the interiors of n disjoint subdisks, and h be

a homeomorphism in $H^n(M_n)$. Then h is B-isotopic to a product of A-homeomorphisms of M_n.

Lemma 2.3. Let M_n be a manifold defined in Lemma 2.2 and h be a homeomorphism in $H^n(M_n)$ where $n \geq 3$. Then up to an isotopy in $H^n(M_n)$, there is a unique homeomorphism g in $H^n(M_n)$ such that $h \approx g$ in $H_n^+(M_n)$ and g is a product of A-homeomorphisms supported on the annuli enclosing more than one boundary hole.

Proof: Let h be a homeomorphism in $H^n(M_n)$. Then h is B-isotopic to a product of A-homeomorphisms by Lemma 2.2. We note that each annulus A_i $(1 \leq i \leq n)$ can be chosen so close to the boundary curve C_i as to be disjoint from the other annuli defining the above A-homeomorphisms.

We first observe that the lemma follows trivially for the case $n = 3$. Since all the possible annuli in $\mathrm{Int}(M_3)$ can be regarded as the annuli A_i $(1 \leq i \leq 3)$ enclosing only one corresponding hole C_i with $A_i \cap A_j = \phi$ for $1 \leq i \neq j \leq 3$, Lemma 2.2 implies that every $h \in H^3(M_3)$ is B-isotopic to a homeomorphism of the form $h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}$ for some integers k_i $(1 \leq i \leq 3)$ and thus the homeomorphism g is in fact the identity.

Now let h be a homeomorphism in $H^n(M_n)$ where $n \geq 4$. Then h is B-isotopic to a product of A-homeomorphisms, some of which may be h_{A_i} for $1 \leq i \leq n$. But since each annulus A_i $(1 \leq i \leq n)$ can be taken to be disjoint from the other annuli defining the A-homeomorphism, each h_{A_i} can be deformed to the identity in $H_n^+(M_n)$ by Lemma 2.1. Thus in the product of A-homeomorphisms, all

h_{A_i} $(1 \leq i \leq n)$ have been eliminated and the homeomorphism g is obtained. The uniqueness of such homeomorphism g follows trivially and the proof is complete.#

Theorem 2.4. Let M_n be a manifold defined in Lemma 2.2 and h a homeomorphism in $H^n(M_n)$. Then in the product of A-homeomorphisms which is B-isotopic to h, the exponent of the homeomorphism h_{A_i} is unique for $1 \leq i \leq n$.

Proof: We assume $n > 2$ since otherwise every $h \in H^2(M_2)$ is B-isotopic to a homeomorphism of the form $h^k_{A_1}$ (or equivalently $h^k_{A_2}$) and the theorem follows trivially.

By Lemma 2.2 h is B-isotopic to a product of A-homeomorphisms. Without loss of generality we assume that h is B-isotopic to two different products of the following forms;

$$h_{A_1}^{k_1} h_{A_2}^{k_2} \cdots h_{A_n}^{k_n} g \qquad \text{(A)}$$

and

$$h_{A_1}^{\ell_1} h_{A_2}^{\ell_2} \cdots h_{A_n}^{\ell_n} g', \qquad \text{(B)}$$

with $k_1 \neq \ell_1$, where k_i and ℓ_i $(1 \leq i \leq n)$ are integers and g and g' are products of A-homeomorphisms supported on the annuli enclosing more than one hole and B-isotopic to each other by Lemma 2.3. Since the annuli A_i $(1 \leq i \leq n)$ can be taken to be disjoint from the other annuli enclosing more than one hole, with $A_i \cap A_j = \phi$ for $1 \leq i \neq j \leq n$, the homeomorphisms h_{A_i} commute with one another and with the homeomorphisms g and g'. Also we note that as in the proof

of Lemma 2.3, the homeomorphisms g and g' are the identity for the case $n = 3$.

It is enough to consider an arc α connecting two boundary curves C_1 and C_2 with $\mathrm{Int}(\alpha) \subset \mathrm{Int}(M_n)$. Thus from the products (A) and (B), we need to consider only the reduced forms $h_{A_1}^{k_1} h_{A_2}^{k_2} g$ and $h_{A_1}^{\ell_1} h_{A_2}^{\ell_2} g'$, since A_i $(3 \leq i \leq n)$ can be taken to be disjoint from the arc α and the annuli enclosing more than one hole and thus

$$h_{A_1}^{k_1} h_{A_2}^{k_2} g(\alpha) = h_{A_1}^{k_1} h_{A_2}^{k_2} \cdots\cdots h_{A_n}^{k_n} g(\alpha)$$

and

$$h_{A_1}^{\ell_1} h_{A_2}^{\ell_2} g'(\alpha) = h_{A_1}^{\ell_1} h_{A_2}^{\ell_2} \cdots\cdots h_{A_n}^{\ell_n} g'(\alpha).$$

We denote $a_i = C_i \cap \alpha$ and $b_i = B_i \cap \alpha$ for $i = 1$ and 2 where B_i is the external boundary curve of A_i. Let $\alpha_1 = [a_1, b_1]$, $\alpha_2 = [b_2, a_2]$ and $\beta = [b_1, b_2]$.

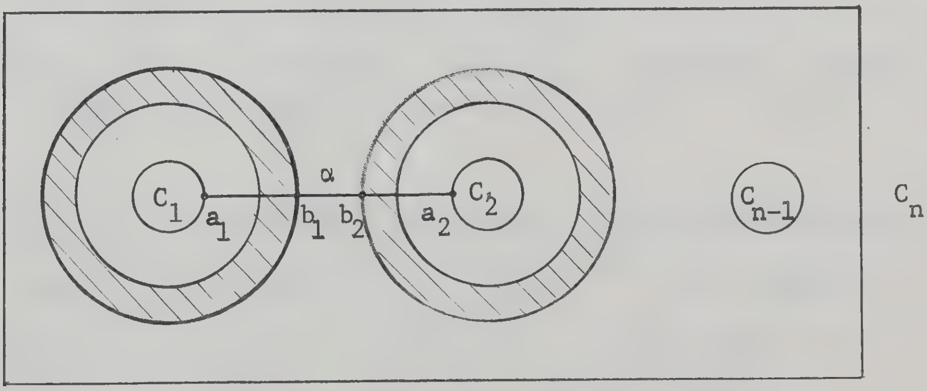

Figure 3

Since

$$[h_{A_1}^{k_1}(\alpha_1)] \cup [h_{A_2}^{k_2}(\alpha_2)] = h_{A_1}^{k_1}h_{A_2}^{k_2}g(\alpha_1 \cup \alpha_2)$$

and
$$[h_{A_1}^{\ell_1}(\alpha_1)] \cup [h_{A_2}^{\ell_2}(\alpha_2)] = h_{A_1}^{\ell_1}h_{A_2}^{\ell_2}g'(\alpha_1 \cup \alpha_2),$$

we see that

$$h_{A_1}^{k_1}h_{A_2}^{k_2}g(\alpha) = [h_{A_1}^{k_1}(\alpha_1)] \circ [g(\beta)] \circ [h_{A_2}^{k_2}(\alpha_2)]$$

and
$$h_{A_1}^{\ell_1}h_{A_2}^{\ell_2}g'(\alpha) = [h_{A_1}^{\ell_1}(\alpha_1)] \circ [g'(\beta)] \circ [h_{A_2}^{\ell_2}(\alpha_2)].$$

Now since g is B-isotopic to g', we have

$$(h_{A_1}^{k_1}h_{A_2}^{k_2}g)(h_{A_1}^{\ell_1}h_{A_2}^{\ell_2}g')^{-1}(\alpha) \simeq [h_{A_1}^{(k_1-\ell_1)}(\alpha_1)] \circ \beta \circ [h_{A_2}^{(k_2-\ell_2)}(\alpha_2)]$$

Then

$$\{[h_{A_1}^{(k_1-\ell_1)}(\alpha_1)] \circ \beta \circ [h_{A_2}^{(k_2-\ell_2)}(\alpha_2)]\} \circ \alpha^{-1} \simeq \omega_1^{(k_1-\ell_1)} \circ \omega_2^{(k_2-\ell_2)},$$

where ω_1 and ω_2 are different generators of the homotopy group $\pi_1(M_n,x_o)$ with $x_o = a_1$. But $\omega_1^{(k_1-\ell_1)} \circ \omega_2^{(k_2-\ell_2)} \neq 0$ unless $k_1 - \ell_1 = 0$ and $k_2 - \ell_2 = 0$. Thus $k_1 \neq \ell_1$ implies that

$$h_{A_1}^{k_1}h_{A_2}^{k_2}g(\alpha) \neq h_{A_1}^{\ell_1}h_{A_2}^{\ell_2}g'(\alpha)$$

and $$h_{A_1}^{k_1} h_{A_2}^{k_2} \cdots\cdots h_{A_n}^{k_n} g(\alpha) \neq h_{A_1}^{\ell_1} h_{A_2}^{\ell_2} \cdots\cdots h_{A_n}^{\ell_n} g'(\alpha).$$

Then Theorem 1.8 implies that

$$h_{A_1}^{k_1} h_{A_2}^{k_2} \cdots\cdots h_{A_n}^{k_n} g \neq h_{A_1}^{\ell_1} h_{A_2}^{\ell_2} \cdots\cdots h_{A_n}^{\ell_n} g'$$

which is a contradiction. Thus $k_1 = \ell_1$ and the exponent of h_{A_1} must be unique in the product. Similarly it can be shown that the exponents of the other homeomorphisms h_{A_i} are also unique for $2 \leq i \leq n$.#

Definition 2.5. Let M_n be a manifold obtained from the 2-sphere by removing the interiors of n disjoint subdisks and h be a homeomorphism in $H^n(M_n)$. Suppose that h' is a product of A-homeomorphisms which is B-isotopic to h. The winding number of h around each of the boundary holes C_i is defined to be the exponent of the homeomorphism h_{A_i} in the product h'. If h_{A_i} does not appear in the product h', the winding number of h around the boundary hole C_i is defined to be zero. We denote by $W[h;C_i]$ the winding number of h around the boundary hole C_i.

Remark 2.6. If h is a homeomorphism in $H^t_{n-t}(M_n)$, then h can be deformed to a homeomorphism $g_1 \in H^n(M_n)$ by Lemma 1.13, where g_1 is a product of A-homeomorphisms including h_{A_i} $(1 \leq i \leq n)$. But by Lemma 2.1 each of the homeomorphisms h_{A_i}, supported around the corresponding hole C_i in the set of $n-t$ holes, is isotopic to the

identity in $H^t_{n-t}(M_n)$. Thus g_1 is further deformed to a homeomorphism $g_2 \in H^n(M_n)$ by an isotopy path in $H^t_{n-t}(M_n)$ such that $W[g_2;C_i] = 0$ for the $n - t$ boundary holes C_i. The winding number of $h \in H^t_{n-t}(M_n)$ around each of the t boundary holes C_j is defined to be $W[g_2;C_j]$.

In what follows M_3 will denote the manifold obtained from the 2-sphere by removing the interiors of three disjoint subdisks.

Theorem 2.7. The isotopy group of $H^3(M_3)$ is $J \times J \times J$.

Proof: We note that every $h \in H^3(M_3)$ is B-isotopic to a product of A-homeomorphisms and as in the proof of Lemma 2.3, h is B-isotopic to a homeomorphism of the form $h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}$ for some integers k_i $(1 \leq i \leq 3)$. Thus we have $\mathscr{I}[H^3(M_3)] \cong \mathscr{I}[\{h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}\}]$ where by Theorem 2.4 any two homeomorphisms of the form $h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}$ having different exponents of h_{A_i} are not B-isotopic to each other. But each h_{A_i} generates the isotopy group J classified by the winding numbers $W[h_{A_i}^{k_i};A_i]$. Since $A_i \cap A_j = \phi$ for $1 \leq i \neq j \leq 3$, the homeomorphisms h_{A_i} $(1 \leq i \leq 3)$ commute with one another. Thus the isotropy group $\mathscr{I}[\{h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}\}]$ is $J \times J \times J$, and the theorem follows.#

Lemma 2.8. If a homomorphism ϕ of a group G onto a group G' induces an isomorphism with G' of a normal subgroup N of G, then $G \cong \ker \phi \times N$.

Proof: This is Theorem 3 on page 112 of [25].

Theorem 2.9. The homeotopy group $\mathscr{H}(M_3)$ is $S_3 \times J_2$.

Proof: We first observe that every $h \in H_3^+(M_3)$ is isotopic to the identity in $H_3^+(M_3)$. By Lemma 1.13 h is isotopic to a homeomorphism $g \in H^3(M_3)$, and g is B-isotopic to a homeomorphism of the form $h_{A_1}^{k_1} h_{A_2}^{k_2} h_{A_3}^{k_3}$ for some integers k_i $(1 \leq i \leq 3)$. But by Lemma 2.1 each $h_{A_i}^{k_i}$ can be deformed to the identity by rotating the boundary curve C_i through 0 to $2(-k_i)\pi$. Hence g (and thus h) is isotopic to the identity in $H_3^+(M_3)$ and thus the homeotopy group is

$$\frac{H(M_3)}{H_3^+(M_3)}.$$

To compute the homeotopy group, we note that

$$H^+(M_3) - H_3^+(M_3) = \{h \in H^+(M_3) \mid h(C_i) \neq C_i \text{ for some } i\}$$

and express the representative homeomorphism of each isotopy class in $H^+(M_3)$ in terms of a linear transformation. Without loss of generality we consider the unit 2-sphere where the boundary holes have centers $(1,0,0)$, $(\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $(\frac{-1}{2}, \frac{-\sqrt{3}}{2}, 0)$ with the same small radius. In Figure 3 we denote ℓ_1 = line passing through $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $(\frac{-1}{2}, \frac{-\sqrt{3}}{2}, 0)$, ℓ_2 = line passing through $(\frac{1}{2}, \frac{-\sqrt{3}}{2}, 0)$ and $(\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0)$.

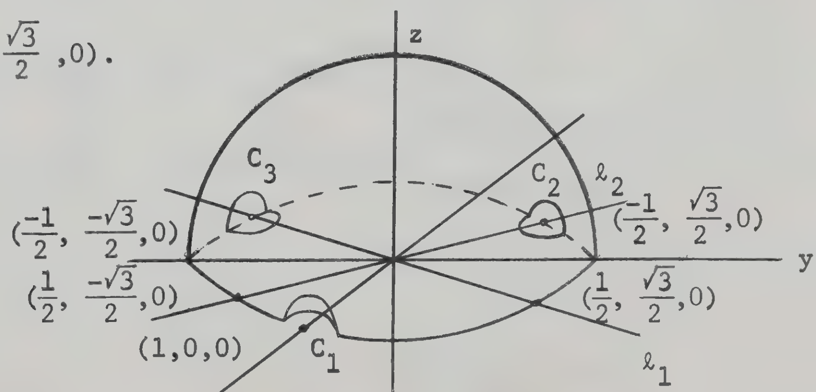

Figure 4

In $H^{+}(M_3)$ there are six representative homeomorphisms as follows.

$$T_1 = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = (C_1C_2C_3);$$ Rotation through 120° around z-axis

$$T_2 = \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} = (C_1C_2)(C_3);$$ Rotation through 180° around axis ℓ_1

$$T_3 = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} = (C_1C_3)(C_2);$$ Rotation through 180° around axis ℓ_2

$$T_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (C_2C_3)(C_1);$$ Rotation through 180° around x-axis

$$T_5 = \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = (C_1C_3C_2);$$ Rotation through 240° around z-axis

$$T_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (C_1)(C_2)(C_3); \text{ The identity.}$$

Observing that T_1 and T_2 generate the other four homeomorphisms T_i $(3 \leq i \leq 6)$;

$$T_3 = T_1T_2, \quad T_4 = T_2T_1, \quad T_5 = T_1^2 \text{ and } T_6 = T_2^2,$$

we see that $T = \{T_i \mid 1 \leq i \leq 6\}$ is a group S_3, the symmetric group on three letters. Hence

$$\frac{H^+(M_3)}{H_3^+(M_3)} = \{T_i[H_3^+(M_3)] \mid 1 \leq i \leq 6\} \cong S_3.$$

Further we note that $\dfrac{H_3(M_3)}{H_3^+(M_3)} \cong J_2$ by Theorem 1.17, and $\dfrac{H^+(M_3)}{H_3^+(M_3)}$ and $\dfrac{H_3(M_3)}{H_3^+(M_3)}$ are normal subgroups of the homeotopy group $\mathscr{H}(M_3)$ since $H^+(M_3)$ and $H_3(M_3)$ are normal subgroups of $H(M_3)$.

Now we show that $\mathscr{H}(M_3) \cong \dfrac{H^+(M_3)}{H_3^+(M_3)} \times \dfrac{H_3(M_3)}{H_3^+(M_3)}$. Let ϕ be the canonical homomorphism from $\dfrac{H(M_3)}{H_3^+(M_3)}$ onto $\dfrac{H(M_3)/H_3^+(M_3)}{H^+(M_3)/H_3^+(M_3)}$, where the range is isomorphic to $\dfrac{H(M_3)}{H^+(M_3)} \cong J_2$. Then it can be seen that the image of $\dfrac{H_3(M_3)}{H_3^+(M_3)}$ under ϕ contains two different elements. Thus ϕ induces an isomorphism of $\dfrac{H_3(M_3)}{H_3^+(M_3)}$ onto its range. Hence by

applying Lemma 2.8, we obtain

$$\mathcal{H}(M_3) = \frac{H(M_3)}{H_3^+(M_3)} \cong \frac{H^+(M_3)}{H_3^+(M_3)} \times \frac{H_3(M_3)}{H_3^+(M_3)} \cong S_3 \times J_2. \#$$

Theorem 2.10. The isotopy group of $H_1(M_3)$ is $J_2 \times J_2$, where $H_1(M_3) = \{h \varepsilon H(M_3) \mid h(C_1)=C_1\}$.

Proof: We note that $\mathcal{J}[H_3^+(M_3)] = [e]$ in $\mathcal{J}[H_1(M_3)]$, since as in the proof of Theorem 2.9, every $h \varepsilon H_3^+(M_3)$ is isotopic to the identity in $H_3^+(M_3)$ and $H_3^+(M_3) \subset H_1(M_3) \subset H(M_3)$. Thus it is enough to calculate the isotopy classes of the subspace $H_1(M_3) - H_3^+(M_3)$. We divide $H_1(M_3) - H_3^+(M_3) = K_1 \cup K_2 \cup K_3$, where

$$K_1 = \{h \varepsilon H_1^+(M_3) \mid h(C_2)=C_3\}$$

$$K_2 = H_3^-(M_3)$$

and

$$K_3 = \{h \varepsilon H_1^-(M_3) \mid h(C_2)=C_3\}.$$

Then it can be seen that each K_i $(1 \leq i \leq 3)$ is generated by one representative homeomorphism T_i of K_i and homeomorphisms in $H_3^+(M_3)$, since for every $h_i \varepsilon K_i$, $g_i = T_i^{-1} h_i \varepsilon H_3^+(M_3)$ and $h_i = T_i g_i$. Thus we need to consider only the representative homeomorphisms.

We express the representative homeomorphism T_i of each K_i in terms of a linear transformation. Without loss of generality we also consider the unit 2-sphere where the boundary holes have centers $(1,0,0)$, $(\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $(\frac{-1}{2}, \frac{-\sqrt{3}}{2}, 0)$ with the same small radius.

Then each T_i is expressed as follows.

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (C_2C_3)(C_1); \quad \text{Rotation through } 180° \text{ around x-axis}$$

$$T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (C_1)(C_2)(C_3); \text{ Reflection in xy-plane}$$

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (C_2C_3)(C_1); \quad \text{Reflection in xz-plane.}$$

We note that the class K_3 is generated by K_1 and K_2 since $T_3 = T_2T_1$. Hence the isotopy classes of $H_1(M_3) - H_3^+(M_3)$ are generated by the classes of K_1 and K_2, each of which generates a group J_2.

Now we show that the isotopy group $\mathcal{J}[H_1(M_3)]$ is the direct product of the groups generated by the isotopy classes of K_1 and K_2. Let G_i be the group generated by the isotopy class of K_i for $i = 1$ and 2. Then by the above arguments,

$$\mathcal{J}[H_1(M_3)] \cong G_1 \circ G_2 = \{g_1g_2 \mid g_i \varepsilon G_i,\ i=1,2\}.$$

Further observing that $G_1 \cap G_2 = \{[e]\}$ and each G_i is a normal subgroup of $\mathcal{J}[H_1(M_3)]$, we have $\mathcal{J}[H_1(M_3)] \cong G_1 \times G_2$ and thus the isotopy group is $J_2 \times J_2$. #

Theorem 2.11. $\mathscr{I}[H_2(M_3)] = \mathscr{I}[H_3(M_3)] = J_2$.

Proof: This follows from Theorem 1.17 since $H_2(M_3) = H_3(M_3)$.

Theorem 2.12. The isotopy group of $H_2^1(M_3)$ is J, where $H_2^1(M_3) = \{h \varepsilon H^+(M_3) \mid h=e \text{ on } C_1 \text{ and } h(C_i)=C_i \text{ for } i=1,2\}$.

Proof: By Remark 2.6 it can be seen that the isotopy group $\mathscr{I}[H_2^1(M_3)]$ is J classified by the winding numbers $\{W[h;C_1] \mid h \varepsilon H_2^1(M_3)\}$.

Theorem 2.13. The isotopy group of $H^2(M_3)$ is $J \times J$, where $H^2(M_3) = \{h \varepsilon H(M_3) \mid h=e \text{ on } C_1 \cup C_2\}$.

Proof: We first note that $H^2(M_3) \subset H^+(M_3)$ and every $h \varepsilon H^2(M_3)$ can be deformed to a homeomorphism in $H^3(M_3)$ by an isotopy path in $H^2(M_3)$. But the isotopy in $H^2(M_3)$ is allowed to move on the boundary curve C_3. Thus by a similar argument as in the proof of Theorem 2.3, we see that every $h \varepsilon H^2(M_3)$ is isotopic to a homeomorphism of the form $h_{A_1}^{k_1} h_{A_2}^{k_2}$ for some integers k_1 and k_2. Hence

$$\mathscr{I}[H^2(M_3)] \cong \mathscr{I}[\{h_{A_1}^{k_1} h_{A_2}^{k_2} \mid k_i \varepsilon J\}],$$

where any two homeomorphisms of the form $h_{A_1}^{k_1} h_{A_2}^{k_2}$ having different exponents of h_{A_i} are not isotopic to each other in $H^2(M_3)$. Thus we see that $\mathscr{I}[\{h_{A_1}^{k_1} h_{A_2}^{k_2}\}] = J \times J$ and the theorem follows. #

Now we consider the homeotopy and isotopy groups of the orientable 2-manifold with n boundary holes.

Lemma 2.14. Let $H_o(M_n)$ and $H_{no}(M_n)$ be the identity components of $H(M_n)$ and $H_n^+(M_n)$ respectively. Then $H_o(M_n) = H_{no}(M_n)$.

Proof: We note that if $h \approx e$ in $H_n^+(M_n)$, then $h \approx e$ in $H(M_n)$. Thus $H_{no}(M_n) \subset H_o(M_n)$.

Conversely, observe that no homeomorphism in $H^-(M_n)$ or $H^+(M_n) - H_n^+(M_n)$ can belong to $H_o(M_n)$. Thus every $h \in H_o(M_n)$ must fix the boundary curves setwise with orientation preserved and thus belongs to $H_n^+(M_n)$. Further we note that any homeomorphism isotopic to the identity must have its isotopy path in $H_n(M_n)$. Thus every homeomorphism, which is isotopic to the identity in $H(M_n)$, must belong to $H_{no}(M_n)$. Hence $H_{no}(M_n) \supset H_o(M_n)$ and the proof is complete.#

Theorem 2.15. Let M_n be an orientable 2-manifold with n boundary holes. Then the homeotopy group $\mathcal{H}(M_n)$ is homomorphic to $S_n \times J_2$ with the kernel $\mathcal{I}[H_n^+(M_n)]$.

Proof: Let ϕ be the canonical homomorphism from $\dfrac{H(M_n)}{H_{no}(M_n)}$ onto $\dfrac{H(M_n)}{H_n^+(M_n)}$, where by Lemma 2.14 $H_{no}(M_n)$ is the identity component of both $H_n(M_n)$ and $H(M_n)$ and thus $\dfrac{H(M_n)}{H_{no}(M_n)} = \mathcal{H}(M_n)$. Then it can be seen that the kernel of ϕ is $\dfrac{H_n^+(M_n)}{H_{no}(M_n)} = \mathcal{I}[H_n^+(M_n)]$.

We now show that $\dfrac{H(M_n)}{H_n^+(M_n)} \cong S_n \times J_2$. Since $H^+(M_n)$ and $H_n(M_n)$ are normal subgroups of $H(M_n)$, $\dfrac{H^+(M_n)}{H_n^+(M_n)}$ and $\dfrac{H_n(M_n)}{H_n^+(M_n)}$ are also normal subgroups of $\dfrac{H(M_n)}{H_n^+(M_n)}$, where $\dfrac{H^+(M_n)}{H_n^+(M_n)} \cong S_n$ and

$$\frac{H_n(M_n)}{H_n^+(M_n)} \cong \frac{H(M_n)}{H^+(M_n)} \cong \frac{H(M_n)/H_n^+(M_n)}{H^+(M_n)/H_n^+(M_n)} \cong J_2.$$

Let ψ be the canonical homomorphism from $\dfrac{H(M_n)}{H_n^+(M_n)}$ onto $\dfrac{H(M_n)/H_n^+(M_n)}{H^+(M_n)/H_n^+(M_n)}$. Then it can be seen that the image of $\dfrac{H_n(M_n)}{H_n^+(M_n)}$ under ψ contains two difference elements. Thus ψ induces an isomorphism of $\dfrac{H_n(M_n)}{H_n^+(M_n)}$ onto the range of ψ. Thus by applying Lemma 2.8, we obtain

$$\frac{H(M_n)}{H_n^+(M_n)} \cong \frac{H^+(M_n)}{H_n^+(M_n)} \times \frac{H_n(M_n)}{H_n^+(M_n)} \cong S_n \times J_2$$

and the proof is complete.#

<u>Theorem 2.16</u>. Let M_n be an orientable 2-manifold with n boundary holes. Then the isotopy group of $H_m(M_n)$ is homomorphic to $S_{n-m} \times J_2$ with the kernel $\mathcal{J}[H_n^+(M_n)]$.

<u>Proof</u>: Let ϕ be the canonical homomorphism from $\dfrac{H_m(M_n)}{H_{no}(M_n)}$ onto $\dfrac{H_m(M_n)}{H_n^+(M_n)}$. Then the kernel of ϕ is $\dfrac{H_n^+(M_n)}{H_{no}(M_n)} = \mathcal{J}[H_n^+(M_n)]$.

We need to show that $\dfrac{H_m(M_n)}{H_n^+(M_n)} \cong S_{n-m} \times J_2$. Since $H_m^+(M_n)$ and $H_n(M_n)$ are normal subgroups of $H_m(M_n)$, $\dfrac{H_m^+(M_n)}{H_n^+(M_n)}$ and $\dfrac{H_n(M_n)}{H_n^+(M_n)}$ are also normal subgroups of $\dfrac{H_m^+(M_n)}{H_n^+(M_n)}$, where $\dfrac{H_m^+(M_n)}{H_n^+(M_n)} \cong S_{n-m}$ and $\dfrac{H_n(M_n)}{H_n^+(M_n)} \cong \dfrac{H_m(M_n)}{H_m^+(M_n)} \cong \dfrac{H_m(M_n)/H_n^+(M_n)}{H_m^+(M_n)/H_n^+(M_n)} \cong J_2$. Now let ψ be the canonical

homomorphism from $\dfrac{H_m(M_n)}{H_n^+(M_n)}$ onto $\dfrac{H_m(M_n)/H_n^+(M_n)}{H_m^+(M_n)/H_n^+(M_n)}$. Then by a similar argument as in the proof of Theorem 2.15, we can see that ψ induces an isomorphism of the normal subgroup $\dfrac{H_n(M_n)}{H_n^+(M_n)}$ onto the range of ψ. Thus we have

$$\frac{H_m(M_n)}{H_n^+(M_n)} \cong \frac{H_m^+(M_n)}{H_n^+(M_n)} \times \frac{H_n(M_n)}{H_n^+(M_n)} \cong S_{n-m} \times J_2.$$

This completes the proof.

<u>Theorem 2.17</u>. Let M_n be an orientable 2-manifold with n boundary holes. Then the isotopy group of $H_m^t(M_n)$ is homomorphic to the symmetric group $S_{n-(m+t)}$ with the kernel $\mathcal{J}[H_{n-t}^t(M_n)]$.

<u>Proof</u>: Let ϕ be the canonical homomorphism from $\dfrac{H_m^t(M_n)}{H_{(n-t)o}^t(M_n)}$ onto $\dfrac{H_m^t(M_n)}{H_{n-t}^t(M_n)}$, where $H_{(n-t)o}^t(M_n)$ is the identity component of $H_{(n-t)}^t(M_n)$. Then the kernel of ϕ is $\dfrac{H_{n-t}^t(M_n)}{H_{(n-t)o}^t(M_n)} = \mathcal{J}[H_{n-t}^t(M_n)]$. Also we can see that $\dfrac{H_m^t(M_n)}{H_{n-t}^t(M_n)} \cong S_{n-(m+t)}$ and the proof is complete.#

CHAPTER III

AUTOMORPHISMS OF THE FUNDAMENTAL GROUP

In this chapter we consider homeomorphisms of the manifold M_n obtained from the 2-sphere by removing the interiors of n disjoint subdisks. We denote by $\mathcal{A}[\pi_1(M_n,x_o)]$ the group of automorphisms of the homotopy group $\pi_1(M_n,x_o)$. A homeomorphism $h \in H(M_n,x_o)$, the isotropy group at the point $x_o \in M_n$, induces an automorphism $h_\#$ in $\mathcal{A}[\pi_1(M_n,x_o)]$. Thus $H(M_n,x_o)$ has a representation $\alpha : h \to h_\#$ as a group of automorphisms of $\pi_1(M_n,x_o)$. Furthermore, if $h \in H_o(M_n,x_o)$, the identity component of $H(M_n,x_o)$, then an arc from h to the identity in $H(M_n,x_o)$ provides a homotopy of $h(\gamma)$ with γ where $[\gamma] \in \pi_1(M_n,x_o)$, and hence $h_\#$ is the identity automorphism of $\pi_1(M_n,x_o)$. Thus α induces a representation

$$\alpha_\# : \mathcal{I}[H(M_n,x_o)] \to \mathcal{A}[\pi_1(M_n,x_o)]$$

of the isotopy group $\mathcal{I}[H(M_n,x_o)]$ as a subgroup of automorphisms of $\pi_1(M_n,x_o)$.

Thus we can define a homomorphism

$$\phi : \mathcal{I}[H(M_n,x_o)] \to \mathcal{A}[\pi_1(M_n,x_o)]$$

by $\phi([h]) = h_\# \in \mathcal{A}[\pi_1(M_n,x_o)]$, where $h_\#(\gamma) = [h(\gamma)] \in \pi_1(M_n,x_o)$ for any γ in a homotopy class $[\gamma] \in \pi_1(M_n,x_o)$. The domain of the homomorphism ϕ will be taken to be the isotopy groups of various subspaces of $H^+(M_n,x_o)$. Throughout this chapter we assume that the

base point x_o is on the boundary curve C_n and $H^1_{n-1}(M_n) = \{h\varepsilon H^+_n(M_n) \mid h=e \text{ on } C_n\}$. By using the homomorphism ϕ, we describe the isotopy groups of certain subspaces of $H(M_n)$.

Theorem 3.1. Let h be a homeomorphism in $H^n(M_n)$. Then h induces the identity automorphism $h_\# = e_\#$ of the homotopy group $\pi_1(M_n,x_o)$ if and only if h is B-isotopic to a product of the homeomorphisms h_{A_i} $(1 \leq i \leq n-1)$.

Proof: For the $n-1$ generators of $\pi_1(M_n,x_o)$, we take the closed paths γ_i $(1 \leq i \leq n-1)$ which are obtained by tracing the arcs $\overline{x_o a_i}$, $\overline{a_i a_i'}$, $\overline{a_i' b_i'}$, $\overline{b_i' b_i}$ and $\overline{b_i x_o}$ with $\gamma_i \cap \gamma_j \cap [\mathrm{Int}(M_n)] = \phi$ for $i \neq j$ as in Figure 5.

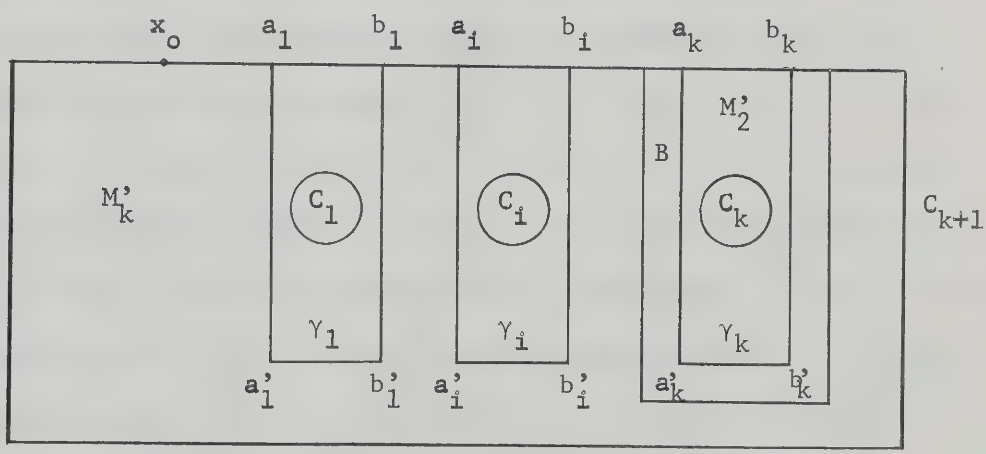

Figure 5

Now assuming that h is B-isotopic to a product of h_{A_i} $(1 \leq i \leq n-1)$, we show that $h_\# = e_\# \varepsilon \mathcal{A}[\pi_1(M_n,x_o)]$. It is sufficient to consider the product of the homeomorphisms h_{A_i}. Define the annuli A_i around the corresponding boundary holes C_i for $1 \leq i \leq n-1$ so small that the annuli A_i do not meet any of the generators γ_i and $\mathrm{Bd}(M_n)$. Then it is clear that

$h_{A_i}(\gamma_j) = \gamma_j$ for $1 \leq i, j \leq n-1$ and thus the product induces the identity automorphism. Thus $h_\# = e_\#$ in $\mathcal{A}[\pi_1(M_n, x_o)]$.

Conversely, letting h be a homeomorphism in $H^n(M_n)$ such that $h_\# = e_\#$, we show by an induction on n that h is B-isotopic to a product of h_{A_i} $(1 \leq i \leq n-1)$. We first note that the theorem follows trivially for the case $n = 2$. Now assuming that our theorem is true for $n = k$, we prove it for $n = k+1$ where the base point x_o is assumed to be on the boundary curve C_{k+1} as in Figure 5. By our assumption we have $h(\gamma_i) \simeq \gamma_i$ keeping the parts $\overline{x_o a_i}$ and $\overline{b_i x_o}$ held fixed for $1 \leq i \leq k$. Denote $\gamma'_k = \overline{a_k a'_k} \cup \overline{a'_k b'_k} \cup \overline{b'_k b_k}$ and $\gamma''_k = \gamma'_k \cup \overline{b_k a_k}$. Then $h(\gamma'_k) \simeq \gamma'_k$ keeping the end points a_k and b_k held fixed and thus these two arcs are ambient isotopic by a B-isotopy, so there is an isotopy $G_t : M_{k+1} \to M_{k+1}$, $0 \leq t \leq 1$, such that $G_0 = e$ on M_{k+1} and $G_1^{-1}h = e$ on γ_k. Now let M'_2 be the closed annulus defined by C_k and γ''_k, and $M'_k = (M_{k+1} - M'_2) \cup \gamma'_k$. Then $G_1^{-1}h \mid M'_k$ is a homeomorphism of M'_k such that $G_1^{-1}h \mid M'_k \in H^k(M'_k)$. Now observe that $(G_1^{-1}h)_\#$ is the identity automorphism of $\pi_1(M'_k, x_o)$. We first note that $G_1^{-1}h(\gamma_i) \subset M'_k$ for $1 \leq i \leq k-1$, since each $\gamma_i \subset M'_k$ and $G_1^{-1}h \mid M'_k \in H^k(M'_k)$. Let B be a narrow band around the arc γ'_k such that $B \cap [\gamma_i \cup G_1^{-1}h(\gamma_i)] = \phi$ for $1 \leq i \leq k-1$. Let g be a homeomorphism from $M_{k+1} \cup [\mathrm{Int}(C_k)]$ onto M'_k such that $g(M'_2 \cup B \cup [\mathrm{Int}(C_k)]) = B$ and $g = e$ on $M_{k+1} - (M'_2 \cup B) = M'_k - B$. Then since $\gamma_i \simeq G_1^{-1}h(\gamma_i)$ on $M_{k+1} \cup [\mathrm{Int}(C_k)]$ for $1 \leq i \leq k-1$, we have $g(\gamma_i) \simeq gG_1^{-1}h(\gamma_i)$ on $g(M_{k+1} \cup [\mathrm{Int}(C_k)]) = M'_k$. But since $g = e$ on $M'_k - B$, $\gamma_i \simeq G_1^{-1}h(\gamma_i)$ on M'_k for $1 \leq i \leq k-1$ and hence $(G_1^{-1}h)_\#$ is the identity automorphism of $\pi_1(M'_k, x_o)$. Thus by

our assumption $G_1^{-1}h \mid M_k'$ is B-isotopic to a product of h_{A_i} $(1 \leq i \leq k-1)$ on M_k'. On the other hand, $G_1^{-1}h \mid M_2'$ is also a homeomorphism of M_2' such that $G_1^{-1}h \mid M_2' \in H^2(M_2')$. Thus $G_1^{-1}h \mid M_2'$ is B-isotopic to a homeomorphism supported on an annulus A_k around the boundary hole C_k such that $A_k \subset \mathrm{Int}(M_2')$. Hence $G_1^{-1}h$ is a homeomorphism of $M_{k+1} = M_k' \cup M_2'$ which is B-isotopic to a product of h_{A_i} $(1 \leq i \leq k)$ on M_{k+1}. But since $G_1^{-1}h$ is B-isotopic to h by Lemma 1.4, h is also B-isotopic to a product of h_{A_i} $(1 \leq i \leq k)$. Thus the theorem is proved for any integer $n \geq 1$. #

Theorem 3.2. The kernel of the homomorphism ϕ is J^{n-1}, if the domain of ϕ is taken to be $\mathcal{J}[H^n(M_n)]$.

Proof: By Theorem 3.1 we know that only the homeomorphisms h, which are B-isotopic to the products of the homeomorphisms h_{A_i} $(1 \leq i \leq n-1)$, induce the identity automorphism of $\pi_1(M_n, x_o)$. But each h_{A_i} generates the isotopy classes J classified by the winding number $W[h;C_i]$ for $1 \leq i \leq n-1$, and thus the kernel of the homomorphism ϕ is J^{n-1}. #

Corollary 3.3. Let h be a homeomorphism in $H^n(M_n)$ which induces the identity automorphism of $\pi_1(M_n, x_o)$. Then h is isotopic to the identity in $H^1_{n-1}(M_n)$.

Proof: By Theorem 3.1, h is B-isotopic to a product of the homeomorphisms h_{A_i} $(1 \leq i \leq n-1)$ of M_n. But by Lemma 2.1, each h_{A_i} in the B-isotopic product of h can be deformed to the identity by rotating the corresponding boundary curve C_i through 0 to $2(-m_i)\pi$ where $m_i = W[h;C_i]$ for $1 \leq i \leq n-1$. Thus h is isotopic to the identity in $H^1_{n-1}(M_n)$. #

Corollary 3.4. Let h be a homeomorphism in $H^n(M_n)$. Then h is isotopic to the identity in $H^1_{n-1}(M_n)$ if and only if h is B-isotopic to a product of the homeomorphisms h_{A_i} $(1 \leq i \leq n - 1)$.

Proof: Assume that h is B-isotopic to a product of the homeomorphisms h_{A_i} $(1 \leq i \leq n - 1)$. Then Theorem 3.1 implies that h induces the identity automorphism of $\pi_1(M_n,x_o)$. Thus by Corollary 3.3, we can see that h is isotopic to the identity in $H^1_{n-1}(M_n)$.

Conversely, since h is isotopic to the identity, an arc from h to the identity in $H^1_{n-1}(M_n)$ provides a homotopy of $h(\gamma)$ with γ for any loop γ in $[\gamma] \in \pi_1(M_n,x_o)$. Thus we have $h_{\#} = e_{\#}$ in $\mathcal{A}[\pi_1(M_n,x_o)]$ and the corollary is proved by Theorem 3.1.$_{\#}$

Theorem 3.5. The homomorphism ϕ is an isomorphism of $\mathcal{J}[H^1_{n-1}(M_n)]$ onto $\phi(\mathcal{J}[H^1_{n-1}(M_n)]) \subset \mathcal{A}[\pi_1(M_n,x_o)]$.

Proof: We need to show that the kernel of ϕ is the isotopy class $[e]$ in $\mathcal{J}[H^1_{n-1}(M_n)]$. By Theorem 3.1 it is clear that the kernel of ϕ is the collection of the classes $[h]$ of the homeomorphisms h which are isotopic to the products of the homeomorphisms h_{A_i} $(1 \leq i \leq n - 1)$ by the isotopy paths in $H^1_{n-1}(M_n)$. But by Corollary 3.4, any of such products can be deformed to the identity and hence every h such that $[h] \in \ker \phi$ is isotopic to the identity in $H^1_{n-1}(M_n)$. Thus $\ker \phi = \{[e]\} \subset \mathcal{J}[H^1_{n-1}(M_n)]$ and the homomorphism ϕ is a monomorphism, which implies that ϕ is an isomorphism onto $\phi(\mathcal{J}[H^1_{n-1}(M_n)])$.$_{\#}$

Theorem 3.6. For the two different domains $\mathcal{J}[H^n(M_n)]$ and $\mathcal{J}[H^1_{n-1}(M_n)]$, the homomorphism ϕ induces an isomorphism;

$\phi(\mathcal{J}[H^n(M_n)]) \cong \phi(\mathcal{J}[H^1_{n-1}(M_n)])$ in $\mathcal{A}[\pi_1(M_n,x_o)]$.

Proof: We observe that

$$\mathcal{J}[H^n(M_n)]/\mathcal{J}[K] \cong \mathcal{J}[H^1_{n-1}(M_n)] \qquad \text{(A)}$$

where K is the collection of the homeomorphisms in $H^n(M_n)$ which are B-isotopic to the products of the homeomorphisms h_{A_i} $(1 \leq i \leq n-1)$. Define a homomorphism

$$\psi : \mathcal{J}[H^n(M_n)] \to \mathcal{J}[H^1_{n-1}(M_n)]$$

by $\psi([h]^*) = [h]_*$, where $[h]^*$ and $[h]_*$ are the isotopy classes of the homeomorphism h in $\mathcal{J}[H^n(M_n)]$ and $\mathcal{J}[H^1_{n-1}(M_n)]$ respectively. Then ψ is an epimorphism and $\ker \psi = \mathcal{J}[K]$ by Corollary 3.4. Thus the relation (A) follows by the isomorphism induced by ψ. But by Theorem 3.1, only the homeomorphisms h in K induce the identity automorphism of the homotopy group $\pi_1(M_n,x_o)$, and thus

$$\mathcal{J}[H^n(M_n)/\mathcal{J}[K] \cong \phi(\mathcal{J}[H^n(M_n)]) \qquad \text{(B)}$$

On the other hand, by Theorem 3.5 we have

$$\mathcal{J}[H^1_{n-1}(M_n)] \cong \phi(\mathcal{J}[H^1_{n-1}(M_n)]) \qquad \text{(C)}$$

Combining the expressions (A), (B) and (C), the theorem is established. #

Lemma 3.7. Let h be a homeomorphism in $H^n(M_n)$. Then h is isotopic

to the identity in $H_n^+(M_n)$ if and only if h is B-isotopic to a product of the homeomorphisms h_{A_i} $(1 \leq i \leq n)$.

Proof: We assume $n > 3$ since otherwise the theorem is trivial. It is sufficient to consider only the products of A-homeomorphisms for all possible annuli A by Lemma 2.2. We note that a product h of h_{A_i} $(1 \leq i \leq n)$ is isotopic to the identity in $H_n^+(M_n)$, since by Lemma 2.1, each h_{A_i} can be deformed to the identity by rotating the corresponding boundary curve C_i through 0 to $2(-m_i)\pi$ where $m_i = W[h;C_i]$.

To prove the converse, let h be a product of A-homeomorphisms of the form

$$h = (h_{A_1}^{k_1} h_{A_2}^{k_2} \cdots\cdots h_{A_{n-1}}^{k_{n-1}} h_{A_n}^{k_n}) \cdot g \ ,$$

where g is a homeomorphism which cannot be factored in such a way as to contain only the homeomorphisms h_{A_i} to some powers. Then by the above arguments, we have

$$h = (h_{A_1}^{k_1} h_{A_2}^{k_2} \cdots\cdots h_{A_{n-1}}^{k_{n-1}} h_{A_n}^{k_n}) \cdot g \approx g$$

in $H_n^+(M_n)$. Thus it is enough to consider only the homeomorphism g. Now assume that $g \approx e$ in $H_n^+(M_n)$. Then the isotopy G_t, $0 \leq t \leq 1$, between g and e in $H_n^+(M_n)$ produces a rotation of the boundary curves, since G_t must keep each of the boundary curves held fixed setwise. Now we construct an isotopy H_t which agrees with G_t on C_n and is the identity outside the annulus A_n for $0 \leq t \leq 1$,

where $H_0 = H_1 = e$. Then $H_t^{-1}G_t$, $0 \leq t \leq 1$, is an isotopy in $H^1_{n-1}(M_n)$ between $H_0^{-1}G_0 = e$ and $H_1^{-1}G_1 = g$. Thus the fact that $g \simeq e$ in $H_n^+(M_n)$ would imply that $g \simeq e$ in $H^1_{n-1}(M_n)$. Corollary 3.4 then implies that g must be a product of the homeomorphism h_{A_i} $(1 \leq i \leq n-1)$. This is a contradiction to our assumption and g (and thus h) is not isotopic to the identity in $H_n^+(M_n)$. Hence we can see that only the products of h_{A_i} $(1 \leq i \leq n)$ are isotopic to the identity in $H_n^+(M_n)$, and the proof is complete.#

Theorem 3.8. For the homomorphism ϕ, we have

$$\mathcal{J}[H_n^+(M_n)] \cong \phi(\mathcal{J}[H^1_{n-1}(M_n)])/J.$$

Proof: As in the proof of Theorem 3.6, we define a similar homomorphism $\psi : \mathcal{J}[H^1_{n-1}(M_n)] \to \mathcal{J}[H_n^+(M_n)]$. Then ψ is an epimorphism and the kernel of ψ is the collection of the classes $[h]$ of the homeomorphisms h isotopic to the products of h_{A_i} $(1 \leq i \leq n)$ by the isotopy paths in $H^1_{n-1}(M_n)$, since $H^1_{n-1}(M_n) \subset H_n^+(M_n)$ and such products are isotopic to the identity in $H_n^+(M_n)$ by Lemma 3.7. The isotopy classes of the kernel of ψ are J classified by the winding numbers $\{W[h;C_n] \mid [h] \varepsilon \ker \psi\}$. Thus we have $\mathcal{J}[H^1_{n-1}(M_n)]/J \cong \mathcal{J}[H_n^+(M_n)]$, and the theorem follows by Theorem 3.5.#

Theorem 3.9. $\mathcal{J}[H^1_{n-1}(M_n)] \cong J \times \mathcal{J}[H_n^+(M_n)]$.

Proof: In the proof of Theorem 3.8, we had $\mathcal{J}[H^1_{n-1}(M_n)]/J \cong \mathcal{J}[H_n^+(M_n)]$. We show that the homomorphism ψ, defined in the proof of Theorem 3.8, induces an isomorphism of a normal subgroup of $\mathcal{J}[H^1_{n-1}(M_n)]$ onto $\mathcal{J}[H_n^+(M_n)]$. Let $N = \{[h] \varepsilon \mathcal{J}[H^1_{n-1}(M_n)] \mid W[h;C_n]=0\}$. Then N is

clearly a normal subgroup of $\mathcal{J}[H^1_{n-1}(M_n)]$. We note that the only difference between the isotopies in the two subspaces $H^+_n(M_n)$ and $H^1_{n-1}(M_n)$ is the deformation on the boundary curve C_n. Every $h \in H^+_n(M_n)$ can be deformed to a homeomorphism h_o by an isotopy path in $H^+_n(M_n)$ such that $h_o = e$ on C_n and $W[h_o;C_n] = 0$ and hence $[h_o] \in N$. Thus the restricted homomorphism $\psi' = \psi \mid N$ is an epimorphism.

Further we observe that ψ' is a monomorphism. By a similar argument as in the proof of Theorem 3.8, the kernel of ψ' is the collection of the isotopy classes $[h]$ in N where the homeomorphisms h are isotopic to the products of h_{A_i} $(1 \leq i \leq n)$ by the isotopy paths in $H^1_{n-1}(M_n)$. But every h such that $[h] \in \ker \psi'$ does not contain h_{A_n} in its isotopic product since $W[h;C_n] = 0$, and thus it is isotopic to the identity in $H^1_{n-1}(M_n)$ by Corollary 3.4. Thus we have $\ker \psi' = [e]$ in N and ψ induces an isomorphism ψ' of the normal subgroup N onto $\mathcal{J}[H^+_n(M_n)]$. Hence the proof is complete by Lemma 2.8.#

Theorem 3.10. $\mathcal{J}[H^+_n(M_n)] \cong \mathcal{J}[H^n(M_n)]/J^n$.

Proof: As in the proof of Theorem 3.6, we define a similar homomorphism $\psi : \mathcal{J}[H^n(M_n)] \to \mathcal{J}[H^+_n(M_n)]$. Then by Lemma 3.7, the kernel of ψ is the collection of the classes $[h]$ of the homeomorphisms h which are B-isotopic to the products of h_{A_i} $(1 \leq i \leq n)$. The isotopy classes of the kernel of ψ are J^n classified by the winding numbers $\{W[h;C_i] \mid [h] \in \ker \psi,\ 1 \leq i \leq n\}$. Thus the homomorphism ψ induces an isomorphism of $\mathcal{J}[H^n(M_n)]/J^n$ onto $\mathcal{J}[H^+_n(M_n)]$.#

Theorem 3.11. $\mathcal{J}[H^n(M_n)] \cong J^n \times \mathcal{J}[H_n^+(M_n)]$.

Proof: Define a normal subgroup N of $\mathcal{J}[H^n(M_n)]$, $N = \{[h] \varepsilon \mathcal{J}[H^n(M_n)] \mid W[h;C_i]=0,\ 1 \le i \le n\}$. We show that the homomorphism ψ, defined in the proof of Theorem 3.10, induces an isomorphism of N onto $\mathcal{J}[H_n^+(M_n)]$. First we note that every $h \varepsilon H_n^+(M_n)$ can be deformed to a homeomorphism h_o such that $h_o = e$ on C_i and $W[h_o;C_i] = 0$ for $1 \le i \le n$ and hence $[h_o] \varepsilon N$. Thus the restricted homomorphism $\psi' = \psi \mid N$ is an epimorphism. Further observe that ψ' is a monomorphism. By Lemma 3.7 the kernel of ψ' is the collection of the isotopy classes $[h]$ in N where the homeomorphisms h are B-isotopic to the products of h_{A_i} $(1 \le i \le n)$. But every h such that $[h] \varepsilon \ker \psi'$ does not contain any h_{A_i} in its B-isotopic product since $W[h;C_i] = 0$ for $1 \le i \le n$, and thus it is in fact B-isotopic to the identity. Thus we have $\ker \psi' = [e]$ in N and ψ' is a monomorphism. Hence the homomorphism ψ induces an isomorphism ψ' of the normal subgroup N onto the isotopy group $\mathcal{J}[H_n^+(M_n)]$ and the theorem follows by Lemma 2.8 and Theorem 3.10.#

Theorem 3.12. $\mathcal{J}[H^n(M_n)] \cong \ker \phi \times \phi(\mathcal{J}[H_{n-1}^1(M_n)])$, where the kernel of ϕ is defined on the domain $\mathcal{J}[H^n(M_n)]$.

Proof:
$$\begin{aligned}
\mathcal{J}[H^n(M_n)] &\cong J^n \times \mathcal{J}[H_n^+(M_n)] && \text{(by Theorem 3.11)}\\
&\cong J^{n-1} \times \{J \times \mathcal{J}[H_n^+(M_n)]\}\\
&\cong J^{n-1} \times \{\mathcal{J}[H_{n-1}^1(M_n)]\} && \text{(by Theorem 3.9)}\\
&\cong J^{n-1} \times \phi(\mathcal{J}[H_{n-1}^1(M_n)]) && \text{(by Theorem 3.5)}\\
&\cong \ker \phi \times \phi(\mathcal{J}[H_{n-1}^1(M_n)]) && \text{(by Theorem 3.2).}
\end{aligned}$$

Theorem 3.13. $\mathcal{J}[H_m^t(M_n)] \cong \mathcal{J}_{H(M_n)}[H_m^t(M_n)] \times J^t$ where $m + t = n$.

Proof: Define a homomorphism

$$\psi : \mathcal{I}[H^t_m(M_n)] \to \mathcal{I}_{H(M_n)}[H^t_m(M_n)]$$

by $\psi([h]^*) = [h]_*$, where $[h]^*$ and $[h]_*$ are the isotopy classes of the homeomorphism h in $\mathcal{I}[H^t_m(M_n)]$ and $\mathcal{I}_{H(M_n)}[H^t_m(M_n)]$ respectively. Then ψ is an epimorphism and the kernel of ψ is the collection of the classes $[h]$ of the homeomorphisms h isotopic to the products of h_{A_i} $(1 \leq i \leq n)$ by the isotopy paths in $H^t_m(M_n)$, since $H^t_m(M_n) \subset H^+_n(M_n)$ and such products are isotopic to the identity in $H^+_n(M_n) \subset H(M_n)$ by Lemma 3.7. We note that $\mathcal{I}_{H^t_m(M_n)}[\ker \psi] = J^t$ classified by the winding numbers $\{W[h;C_i] \mid [h] \varepsilon \ker \psi\}$ for each of the t boundary holes C_i. Thus $\mathcal{I}[H^t_m(M_n)]/J^t \cong \mathcal{I}_{H(M_n)}[H^t_m(M_n)]$.

Now define a normal subgroup N of the isotopy group $\mathcal{I}[H^t_m(M_n)]$, $N = \{[h] \varepsilon \mathcal{I}[H^t_m(M_n)] \mid W[h;C_i]=0$ for the t holes $C_i\}$. We note that every isotopy class $[h]$ in $\mathcal{I}_{H(M_n)}[H^t_m(M_n)]$ contains a homeomorphism $h_o \varepsilon H^t_m(M_n)$ such that $W[h_o;C_i] = 0$ for each of the t boundary holes C_i, since the isotopy in $H(M_n)$ is allowed to rotate each of the boundary curves. But $[h_o]$ belongs to the normal subgroup N and thus the restricted homomorphism $\psi' = \psi \mid N$ is an epimorphism. Further we can see that ψ' is a monomorphism. By the above arguments, the kernel of ψ' is the collection of the classes $[h]$ in N where the homeomorphisms h are isotopic to the products of h_{A_i} $(1 \leq i \leq n)$ by the isotopy paths in $H^t_m(M_n)$. But every h such that $[h] \varepsilon \ker \psi'$ does not contain any h_{A_i} in its isotopic product since $W[h;C_i] = 0$, where the annuli A_i are determined around each of the t boundary holes, and thus it is isotopic to the identity in $H^t_m(M_n)$. Thus $\ker \psi' = [e]$ in N and ψ induces an

isomorphism ψ' of the normal subgroup N onto $\mathcal{J}[H_n^+(M_n)]$. Hence the proof is complete by Lemma 2.8. #

CHAPTER IV

ISOTROPY GROUPS

Let $H(X,x)$ denote the group of homeomorphisms of X which leave the point $x \in X$ fixed. This group is called the isotropy group at x. In this chapter we compute the isotopy groups of various isotropy groups. Further by using the homeotopy exact sequence, we calculate the nth homeotopy groups of a certain isotropy group.

Lemma 4.1 (McCarty [17]). If M is a manifold with or without boundary and $x \in \mathrm{Int}(M)$, then $H(M)$ is a fiber bundle over $\mathrm{Int}(M)$ with projection $P : h \to H(x)$ and fiber $H(M,x)$.

If M is a manifold with or without boundary and $x \in \mathrm{Int}(M)$, the homotopy sequence of the bundle $H(M)$ is exact (Definition 3.2; [17]). Thus with the notation of [13], McCarty obtained the following exact sequence which is called the homeotopy sequence of M,

$$\cdots \xrightarrow{P_*} \pi_{n+1}(M,x) \xrightarrow{d_*} \pi_n[H(M,x)] \xrightarrow{i_*} \pi_n[H(M)] \xrightarrow{P_*} \pi_n(M,x) \to \cdots$$

$$\cdots \xrightarrow{P_*} \pi_1(M,x) \xrightarrow{d_*} \pi_0[H(M,x)] \xrightarrow{i_*} \pi_0[H(M)] \xrightarrow{P_*} 0.$$

Lemma 4.2 (Epstein [6]). Let f_0, f_1 be imbeddings of S^1 in $\mathrm{Int}(M)$ such that $f_0 \simeq f_1 : S^1, * \to M, *$. If $f_0(S^1)$ does not bound a disk or Mobius band in M, then there is an ambient isotopy between f_0 and f_1 keeping the base point fixed and which is fixed

outside a compact subset of Int(M).

In what follows A will denote an annulus which we take as the cylinder $S^1 \times I$ with the notations $C_1 = S^1 \times 0$ and $C_2 = S^1 \times 1$.

Theorem 4.3. The isotopy group of $H(A,a)$ is $J_2 \times J_2$ where $a \in \mathrm{Int}(A)$.

Proof: Let a be the point $(0,\frac{1}{2})$. Let S be the subspace of the homeomorphisms in $H^+(A,a)$ which do not interchange the boundary curves C_1 and C_2. We first show that every $h \in S$ is isotopic to the identity in S. Let γ be the closed arc $S^1 \times \frac{1}{2}$ in A. Then it is clear that $h(\gamma)$ does not bound a disk or Mobius band, and $h(\gamma) \simeq \gamma$ fixing the point a since the homotopy group $\pi_1(A,a) = J$ has only the identity and inverse automorphisms and thus h must induce the identity automorphism of $\pi_1(A,a)$. Thus Lemma 4.2 implies that there is an ambient isotopy $G_t : A, a \to A, a$ $(0 \leq t \leq 1)$ such that $G_0 = e$ and $G_1 = h$ on γ. Denote $A_1 = S^1 \times [0,\frac{1}{2}]$ and $A_2 = S^1 \times [\frac{1}{2},1]$. Since $G_1^{-1}h = e$ on γ, we have $G_1^{-1}h \mid A_i \approx e$ on A_i for $i = 1$ and 2 by Lemma 1.12, where the isotopies are the identity on the closed arc $\gamma = A_1 \cap A_2$ and move only on the boundary curves C_1 and C_2 respectively. Thus $G_1^{-1}h \approx e$ on $A = A_1 \cup A_2$, which implies that h is isotopic to the identity in S.

Now to compute the isotopy group, we note that $H^+(A,a)$ forms a normal subgroup of $H(A,a)$ and

$$H^+(A,a) - S = \{h \in H^+(A,a) \mid h(C_1)=C_2\}.$$

Thus $\mathscr{I}[H^+(A,a)] = H^+(A,a)/S \cong J_2$. Further we note that

$H^{-}(A,a) = f \circ \{H^{+}(A,a)\}$ and thus $H(A,a) = \{f,e\} \circ \{H^{+}(A,a)\}$, where f is a representative homeomorphism in $H^{-}(A,a)$, say a homeomorphism flipping the two boundary curves C_1 and C_2. But it can be seen that $\{[f],[e]\} \cong J_2$ forms a normal subgroup of $\mathcal{J}[H(A,a)]$. Thus we have

$$\mathcal{J}[H(A,a)] \cong \mathcal{J}[H^{+}(A,a)] \times \{[f],[e]\} \cong J_2 \times J_2$$

and the proof is complete.#

Lemma 4.4 (Arens [3]). If X is a locally connected and compact Hausdorff space, then, for $x \in X$, $H(X,x)$ is topologically isomorphic to $H(X-x)$ and thus $\pi_n[H(X,x)] \cong \pi_n[H(X-x)]$ for $n \geq 0$.

Corollary 4.5. The homeotopy group $\mathcal{H}(A-a)$ is $J_2 \times J_2$ where $a \in \mathrm{Int}(A)$.

Proof: This follows from Theorem 4.3 and Lemma 4.4.

Remark 4.6. From the above corollary, it can be seen that if M is a manifold obtained from R^2 by removing the interiors of two disjoint subdisks, then the homeotopy group of M is $J_2 \times J_2$.

Theorem 4.7. The isotopy group of $H_2(A,a)$ is J_2, where $a \in \mathrm{Int}(A)$ and $H_2(A,a) = \{h \in H(A,a) \mid h(C_i)=C_i, i=1,2\}$.

Proof: Noting that no homeomorphism in $H_2(A,a)$ interchanges the boundary curves, we can prove the theorem by a similar argument as in the proof of Theorem 4.3.#

Theorem 4.8. The isotopy group of $H^1(A,a)$ is J, where $a \in \mathrm{Int}(A)$ and $H^1(A,a) = \{h \in H(A,a) \mid h=e \text{ on } C_1\}$.

Proof: Let $a = (0,\frac{1}{2})$ and γ be the closed arc $S^1 \times \frac{1}{2}$ in A. We denote $A_1 = S^1 \times [0,\frac{1}{2}]$ and $A_2 = S^1 \times [\frac{1}{2},1]$. Note that $H^1(A,a) \subset H^+(A,a)$ by Theorem 1.17, since every $h \in H^1(A,a)$ is the identity (and thus orientation preserving) on the boundary curve C_1. Thus by a similar argument as in the proof of Theorem 4.3, for every $h \in H^1(A,a)$ there is a homeomorphism $g \in H^2(A,a)$ such that $g \underset{B}{\simeq} e$ on A and $g^{-1}h = e$ on γ. Lemma 1.12 implies that $g^{-1}h \mid A_2 \simeq e$ on A_2 by an isotopy which is the identity on the closed arc γ and moves on the boundary curve C_2. In A_1, since $g^{-1}h = e$ on $Bd(A_1) = \gamma \cup C_1$, the isotopy classes of the restricted homeomorphisms $\{g^{-1}h \mid A_1\}$, for all $h \in H^1(A,a)$ and the above defined homeomorphisms g, are J classified by the winding numbers $\{W[g^{-1}h \mid A_1;A_1]\}$. But a homeomorphism $g^{-1}h$ such that $W[g^{-1}h \mid A_1;A_1] \neq 0$ cannot be isotopic to the identity on $A = A_1 \cup A_2$, since the isotopy in $H^1(A,a)$ leaves $C_1 \cup \{a\}$ pointwise fixed. Thus the isotopy classes of the collection of all such homeomorphisms $g^{-1}h$ on A are J. But we have the relation $\mathcal{J}[H^1(A,a)] \cong \mathcal{J}[\{g^{-1}h\}]$ and thus the isotopy group is J. #

Theorem 4.9. The isotopy group of $H^2(A,a)$ is $J \times J$, where $a \in Int(A)$ and $H^2(A,a) = \{h \in H(A,a) \mid h=e \text{ on } C_1 \cup C_2\}$.

Proof: Let $a = (0,\frac{1}{2})$ and γ be the closed arc $S^1 \times \frac{1}{2}$ in A. We denote $A_1 = S^1 \times [0,\frac{1}{2}]$ and $A_2 = S^1 \times [\frac{1}{2},1]$. For every $h \in H^2(A,a)$, there is a homeomorphism g such that $g \underset{B}{\simeq} e$ on A and $g^{-1}h = e$ on $\gamma \cup C_1 \cup C_2$ by Lemma 4.2. Then for $i = 1$ and 2, the isotopy classes of the restricted homeomorphisms on A_i, $\{g^{-1}h \mid A_i\}$, for all $h \in H^2(A,a)$ and the above defined homeomorphisms g, are J classified by the winding numbers $\{W[g^{-1}h \mid A_i;A_i]\}$. Thus the isotopy

classes of the collection of all such homeomorphisms $g^{-1}h$ on $A = A_1 \cup A_2$ are $J \times J$. But we have the relation $\mathcal{J}[H^2(A,a)] \cong \mathcal{J}[\{g^{-1}h\}]$ and thus the isotopy group is $J \times J$. #

Theorem 4.10 (Kneser [14]). Let D be a disk. Then

$$\pi_n[H(D)] = \begin{cases} J_2 & \text{if } n = 0 \\ J & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

Theorem 4.11 (Quintas [20]). Let a be a point in the interior of a disk D. Then

$$\pi_n[H(D,a)] = \pi_n[H(D-a)] = \begin{cases} J_2 & \text{if } n = 0 \\ J & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

Lemma 4.12 (McCarty [17]). Let X be a locally connected and compact Hausdorff space, and $H(X)$ be a bundle over X with projection P. Then the subgroup $P_*[\pi_1(H(X))]$ is the center of $\pi_1(X,x)$.

Theorem 4.13. Let D be a disk and a, x be two different points in $\mathrm{Int}(D)$. Then

$$\pi_n[H(D-a,x)] = \begin{cases} \mathcal{J}[H(D-a,x)] = J_2 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1. \end{cases}$$

Proof: For the case $n = 0$, for simplicity we consider $S^1 \times (0,1]$ for $D - a$ and let $x = (0,\frac{1}{2})$ and $C = S^1 \times 1$. We first show that

every $h \in H^{+}(D-a,x)$ is isotopic to the identity in $H^{+}(D-a,x)$. Denote $\gamma = S^1 \times \frac{1}{2}$, $A = S^1 \times [\frac{1}{2},1]$ and $B = S^1 \times (0,\frac{1}{2}]$. Then it can be seen that $h(\gamma)$ does not bound a disk or Mobius band and $h(\gamma) \simeq \gamma$ fixing the point x. Thus by Lemma 4.2, there is an ambient isotopy $G_t : D - a, x \to D - a, x$ $(0 \leq t \leq 1)$ such that $G_0 = e$ and $G_1^{-1}h = e$ on the arc γ. Now observe that the restricted homeomorphism $G_1^{-1}h \mid B$ is isotopic to the identity on B. By Lemma 4.4 we can regard $G_1^{-1}h$ as a homeomorphism of a disk onto itself fixing one interior base point. Thus by Lemma 1.1, the homeomorphism $G_1^{-1}h$ is isotopic to the identity on the disk by an isotopy fixing the base point and the boundary γ pointwise. In A, by Lemma 1.12 it can be seen that the restricted homeomorphism $G_1^{-1}h \mid A$ is also isotopic to the identity on A by an isotopy fixing the closed arc γ pointwise and moving on the boundary C. Thus $G_1^{-1}h$ is isotopic to the identity on $D - a$ by an isotopy fixing the point x, and thus the homeomorphism h is isotopic to the identity in $H^{+}(D - a,x)$. Hence we see that

$$\mathcal{J}[H(D-a,x)] = \frac{H(D-a,x)}{H^{+}(D-a,x)} \cong J_2$$

which completes the proof for the case $n = 0$.

For the case $n \geq 1$ we consider the homeotopy exact sequence,

$$\cdots \xrightarrow{P_*} \pi_{n+1}(D-a,x) \xrightarrow{d_*} \pi_n[H(D-a,x)] \xrightarrow{i_*} \pi_n[H(D-a)] \xrightarrow{P_*} \pi_n(D-a,x) \to \cdots$$

In this sequence, $\pi_n[H(D-a)] = \pi_n[H(D-a,x)]$ for $n \geq 2$ since $\pi_n(D-a,x) = 0$ for $n \geq 2$. Thus by Theorem 4.11, we see that $\pi_n[H(D-a,x)] = 0$ for $n \geq 2$.

For $\pi_1[H(D-a,x)]$ we consider the end of the exact sequence,

$$\cdots \to \pi_2(D-a,x) \to \pi_1[H(D-a,x)] \to \pi_1[H(D-a)] \to \pi_1(D-a,x)$$

$$\to \pi_0[H(D-a,x)] \to \pi_0[H(D-a)] \to 0$$

which is explicitly as follows.

$$\cdots \to 0 \xrightarrow{d_*} \pi_1[H(D-a,x)] \xrightarrow{i_*} J \xrightarrow{P_*} J \xrightarrow{d_*} J_2 \xrightarrow{i_*} J_2 \xrightarrow{P_*} 0.$$

In this sequence, by Lemma 4.12 we know that $P_*[\pi_1(H(D-a))]$ is the center of $\pi_1(D-a,x)$. But since $\pi_1(D-a,x) = J$ is an abelian group, P_* is an epimorphism which implies that it is in fact an isomorphism. Thus $\ker P_* = 0$ and $i_*[\pi_1(H(D-a,x))] = 0$. Hence $\ker i_* = \pi_1[H(D-a,x)]$, and since $d_*[\pi_2(D-a,x)] = 0$, we obtain the result $\pi_1[H(D-a,x)] = 0$. This completes the proof.#

Corollary 4.14. The isotopy group of $H^1(D-a,x)$ is J, where a, $x \in \mathrm{Int}(D)$ and $H^1(D-a,x) = \{h \in H(D-a,x) \mid h = e \text{ on } C\}$.

Proof: The proof is similar to those of Theorem 4.8 and 4.13.#

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